Heaps and applications

K. N. Raghavan
IMSc

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at ISI-BC

A plug for videos of Viennot's recently concluded 19-lecture course at IMSc

Xavier Viennot: Commutations and Heaps of Pieces

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19th January to 16th March

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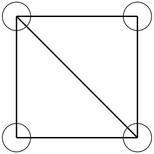
Colourings of a graph



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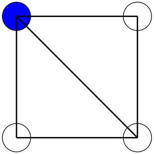

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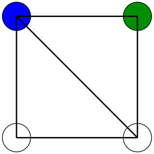

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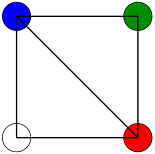

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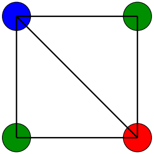

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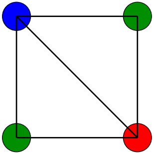

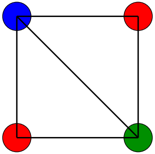

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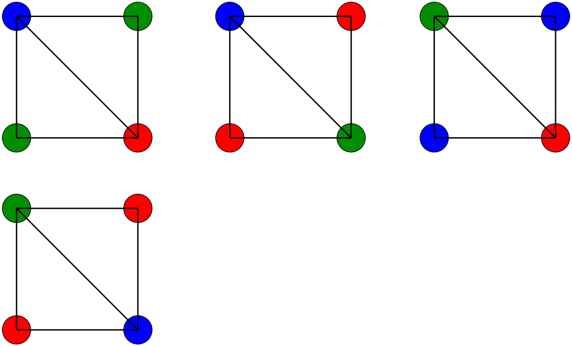

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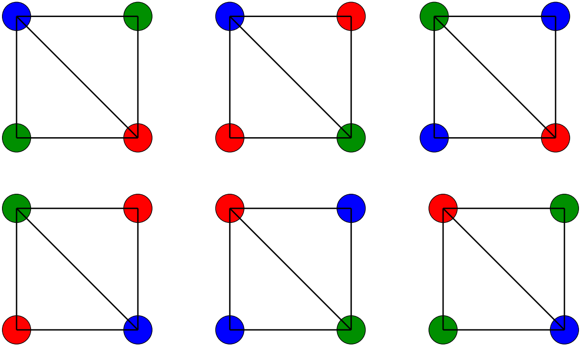

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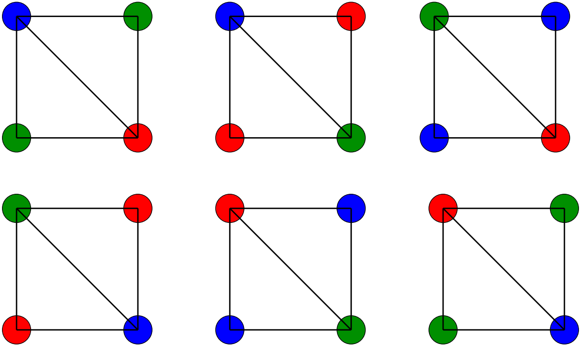

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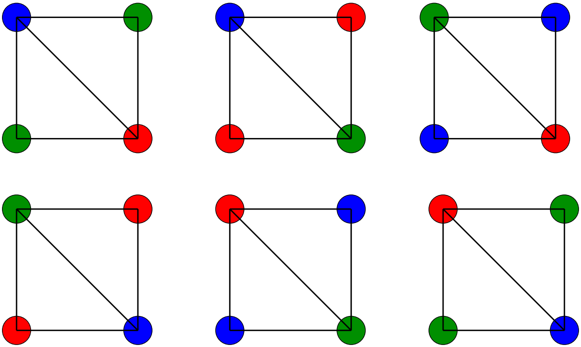

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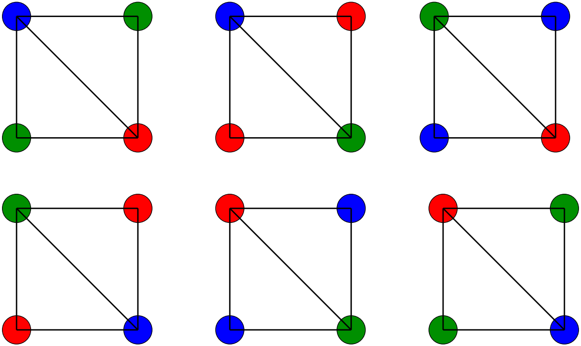

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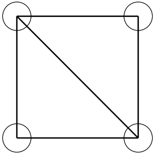

Suppose we are given λ colours with which to colour the graph.

We need to use at least 3 colours and can use at most 4 colours.

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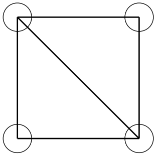

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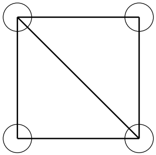

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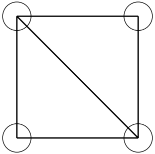

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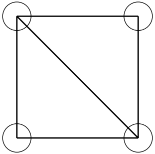

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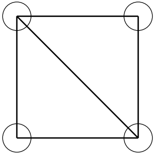

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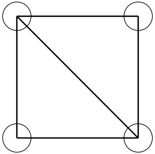

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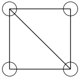
For , we've seen that $\gamma_G(\lambda) = \boxed{6 \times \binom{\lambda}{3} + 24 \times \binom{\lambda}{4}}$

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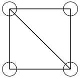
For , we've seen that $\gamma_G(\lambda) = \boxed{6 \times \binom{\lambda}{3} + 24 \times \binom{\lambda}{4}}$

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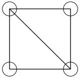
For , we've seen that $\gamma_G(\lambda) = \boxed{6 \times \binom{\lambda}{3} + 24 \times \binom{\lambda}{4}}$

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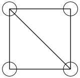
For , we've seen that $\gamma_G(\lambda) = \boxed{6 \times \binom{\lambda}{3} + 24 \times \binom{\lambda}{4}}$

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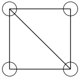
For , we've seen that $\gamma_G(\lambda) = \boxed{6 \times \binom{\lambda}{3} + 24 \times \binom{\lambda}{4}}$

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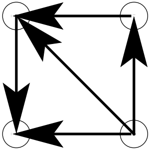

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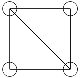
For , we've seen that $\gamma_G(\lambda) = \boxed{6 \times \binom{\lambda}{3} + 24 \times \binom{\lambda}{4}}$

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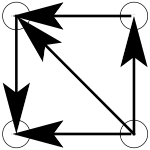

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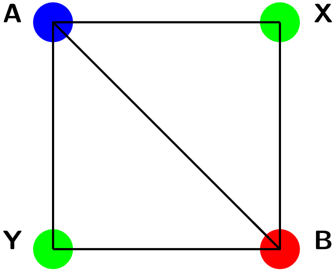

B
XY
A

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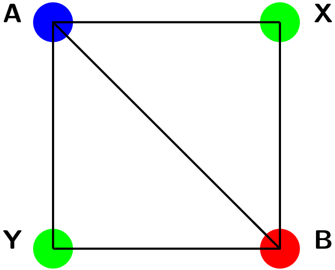

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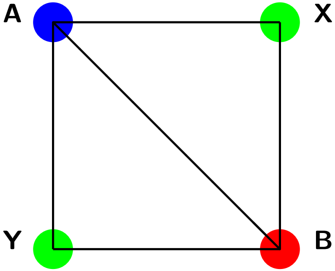

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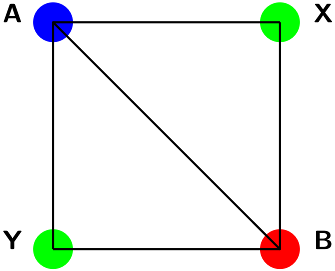

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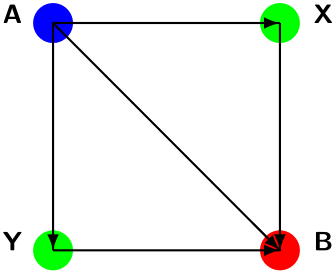
Colourings are layerings and determine acyclic orientations



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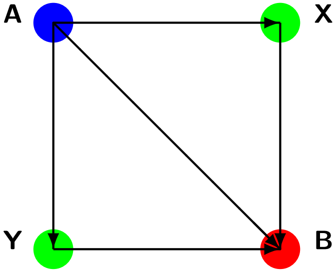
Two colourings may yield the same acyclic orientation



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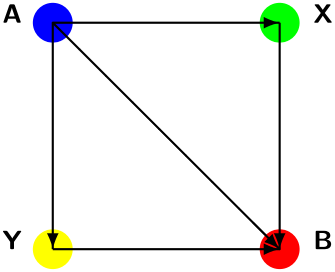

B
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colourings, layerings, acyclic orientations, and ...

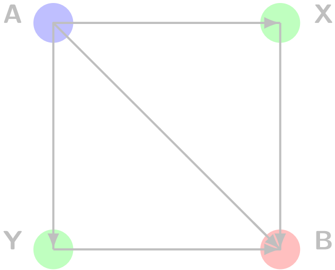
$$\begin{tikzcd}
\text{colourings} \arrow[r, leftrightarrow] \arrow[d] & \text{layerings} \arrow[d] \\
\text{acyclic orientations} \arrow[r, leftrightarrow] & ??
\end{tikzcd}$$

colourings, layerings, acyclic orientations, and **heaps**

$$\begin{array}{ccc}
\text{colourings} & \longleftrightarrow & \text{layerings} \\
\downarrow & & \downarrow \\
\text{acyclic orientations} & \longleftrightarrow & \textbf{(Multilinear) HEAPS}
\end{array}$$

So what are **heaps**?

(Multilinear) Heaps *are layerings in which things fall until obstructed.*



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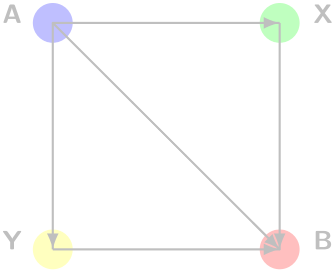

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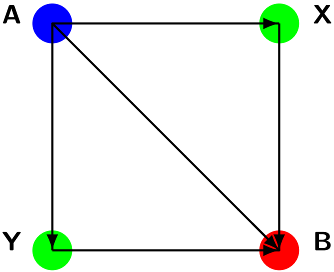

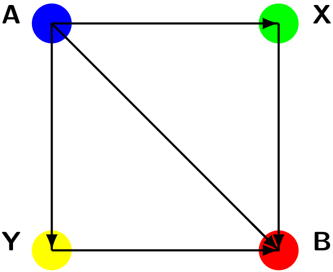

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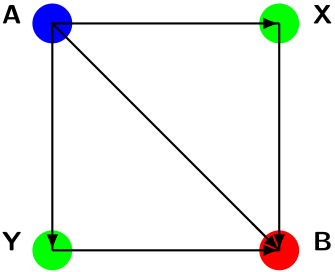

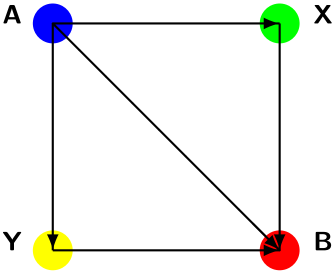

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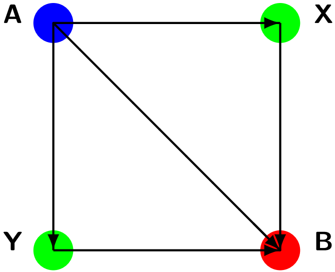

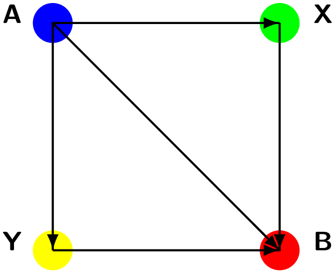

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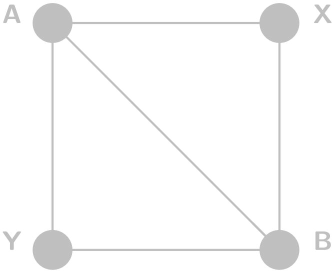

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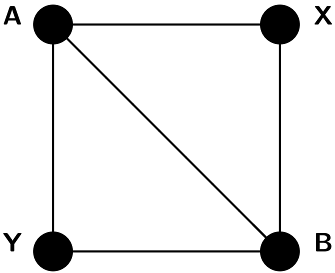

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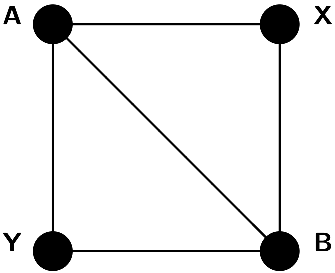

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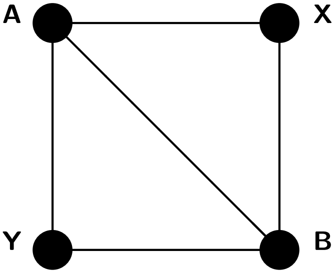

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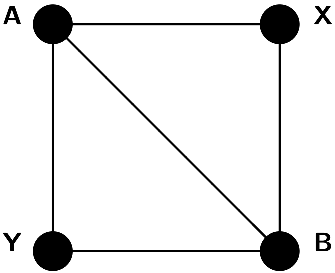

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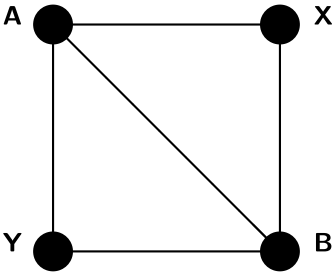

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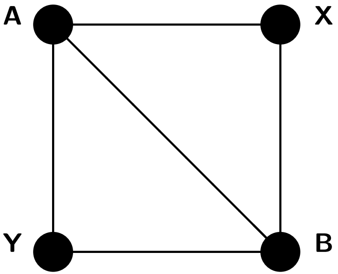

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